ECON0106 Tutorial 1 **2026-2027**

Junzhao Shi

Overview of Section 1

Choice Functions $\leftrightarrow$ Preference Relations

We first consider choice functions $C : 2^X \to 2^X$, which represent how people choose when a bundle of choices is presented to them.

We then define preference relations $\succsim \subseteq X \times X$, which are complete and transitive. Proposition 1 tells us that when specifying completeness and transitivity on $\succsim$, we have additional structures on the asymmetric part $\succ$. Having defined preference relations, Proposition 2 assures us that the argmax has intuitive and desirable properties.

Having defined preference relations, we want a choice function to be represented by a preference relation. Not all choice functions can be represented by a preference relation; only those that satisfy HARP can (Theorem 1).

We can decompose HARP into two intuitive properties:

1. Sen's α (IIA): If one chooses an item from a set, then this item will still be chosen for any subset that this item is still available.
2. Sen's β (expansion consistency): If two items are both chosen in a set, and if one of them is chosen in a superset, then the other item should also be chosen in the super set.

In exercise 4, we show that choice functions that satisfy HARP if and only if they satisfy Sen's α and β.

*Question raised lecture: Are there any examples of choice functions that does not satisfy WARP?

*Answer: Consider 3 alternatives Yogurt, Sorbet and Ice Cream. When presented with Yogurt and Sorbet, the consumer exercise self-control and take Yogurt. When everything is presented, the consumer can only resist unhealthy food to a limited extent and goes for Sorbet as a compromise.

Preference Relations $\leftrightarrow$ Utility Representations

Having shown that we can represent choice functions with preference maximisations, we want to connect preferences with our usual tools of utility maximisation.

By proposition 3, preference relations over a finite number of alternatives admit a utility representation and this representation is unique up to monotone transformation (proposition 4). So do preference relations over countably many alternatives (proposition 5).

It is not possible for preference relations over infinite number of alternatives to admit a utility representation, as there are too many "indifference sets". However, we can regain traction by assuming that the set of alternatives has a countable order-dense subset (theorem 2).

Limited Observability

Since choice functions are mappings from $2^X \to 2^X$, it is impossible to observe people's choices for all menus of alternatives, even if X is finite. If $|X| = 10$, there are 1024 different menus that we need to observe. Hence, we need a stronger assumption on the choice functions, which is GARP, so that we can use preference relations to represent choices (Theorem 5).

Exercise 4

(i) Show that Sen's α is equivalent to the following property: if $B \subseteq A$, then $B \cap C(A) \subseteq C(B)$

Proof. Sen's α $\implies$ the statement:

Take any $B \subseteq A$. For any $x \in B \cap C(A)$, $x \in C(A)$. This implies $x \in C(B)$ by Sen's α.

the statement $\implies$ Sen's α:

Suppose $x \in B \subseteq A \subseteq X$ and $x \in C(A)$. This implies that $x \in C(A) \cap B \subseteq C(B)$, where the subset inclusion is by the statement. □

(ii) Show that Sen's β is equivalent to the following property: if $B \subseteq A$ and $C(A) \cap C(B) \neq \emptyset$, then $C(B) \subseteq C(A)$.

Proof. Sen's β $\implies$ the statement:

Suppose $B \subseteq A \subseteq X$ and $C(A) \cap C(B) \neq \emptyset$. Take any $x \in C(B)$. Since there exists some $y \in C(A) \cap C(B)$. Notice that $x, y \in C(B)$, and $y \in C(A)$. By Sen's β, $x \in C(A)$.

the statement $\implies$ Sen's β:

Suppose $B \subseteq A \subseteq X$, $x, y \in C(B)$ and $y \in C(A)$. Since $y \in C(A) \cap C(B)$, the intersection is non-empty. Therefore, $x \in C(B) \subseteq C(A)$, where the subset inclusion is by the statement. □

(iii) Let $C : 2^X \to 2^X$ be a choice function. Prove that HARP is equivalent to Sen's α and β. Conclude on the properties that $\operatorname{argmax}_{\succsim}$ satisfies, where $\operatorname{argmax}_{\succsim} A := C(A), \forall A \in 2^X$.

Proof. HARP $\implies$ Sen's α and β:

(α). Suppose $B \subseteq A \subseteq X$. Then take any $x \in B \cap C(A)$. Since B is non-empty, there exists some $y \in C(B)$. Since $x, y \in A \cap B$, where $x \in C(A)$ and $y \in C(B)$, then by HARP $x \in C(B)$.

(β). Suppose $B \subseteq A \subseteq X$ and $C(A) \cap C(B) \neq \emptyset$. Take $y \in C(A) \cap C(B) \neq \emptyset$. For any $x \in C(B)$, $x, y \in A \cap B$. By HARP, $x \in C(A)$.

Sen's α and β $\implies$ HARP:

Take any $x, y \in A \cap B$, where $x \in C(A)$ and $y \in C(B)$. Since $A \cap B \subseteq A$, then by Sen's α, $x \in (A \cap B) \cap C(A) \subseteq C(A \cap B)$. Similarly, $y \in C(A \cap B)$. As $A \cap B \subseteq B$ and $y \in (A \cap B) \cap C(A)$, by Sen's β, $x \in C(B)$. By the symmetric argument, $y \in C(A)$. □

Exercise 6

(i) Does this procedure satisfies Sen's α, β?

Both are satisfied.

Proof. Consider $B \subseteq A \subseteq X$.

(α). Suppose there is a book $x \in B$ and $x \in C(A)$. This means that x is either the lowest book in the alphabetical ranking that achieves $\overline{u}$, or it has the highest utility out of all books in A. Getting rid of books that are not x preserves the alphabetical ranking and the x's utility level.

(β). If $x, y \in C(B)$, then there are no books in B that passes utility level $\overline{u}$ and $u(x) = u(y)$. If $y \in C(A)$, this means that there are no books in A that passes utility level $\overline{u}$. Hence $x, y \in C(A)$.

□

(ii) Can the bookseller learn the consumer's preference?

No.

Proof. Consider the case where the consumer is not indifferent between all books, and has a reservation value at $u(z) = \min_x u(x) < \overline{u} < \max_x u(x) = u(y)$. Suppose there is exists a book $x \in C(X)$ where $u(x) \geq \overline{u}$ and $x \neq y$. Notice that for any bundle of books that contain x and y, by Sen's α, the consumer's choice would always be x. □

(iii) If choices are consistent with HARP, we are sure that consumers are choosing their most preferred items.

No. See the counter example above.

Exercise 1* (Section 2)

(i) $\implies$ (ii)

Proof. Fix an x. Take any sequence $y_n \to y$ such that $y_n \succsim x$ for all n. Since $\succsim$ is continuous, there is $y \succsim x$. The other case the same. □

(ii) $\implies$ (iii)

Proof. Since $X_{x\succsim}$ is closed, $X_{\succ x} = X \backslash X_{x\succsim}$ is open. The other case the same. □

(iii) $\implies$ (iv)

Proof. As $X_{\succ y}$ is an open set, there exists some $\epsilon_1 > 0$ such that $B_{\epsilon_1}(x) \subseteq X_{\succ y}$. Consider the intersection $B_{\epsilon_1}(x) \cap X_{x\succ}$. If this intersection is empty, $B_{\epsilon_1}(x) \subseteq X_{\precsim x}$. Because $X_{x\succ}$ is open, there is ϵ_2 such that $B_{\epsilon_2}(y) \subseteq X_{x\succ}$. Set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$.

If the intersection is non-empty, let z be a point in $B_{\epsilon_1}(x) \cap X_{x\succ}$. Then $x \succ z \succ y$. Since $X_{z\succ}$ is also open, there is an ϵ_3 such that $B_{\epsilon_3}(y) \subseteq X_{z\succ}$. Notice that $X_{\succ z}$ is an open set as well. Therefore, there exists an ϵ_4 such that $B_{\epsilon_4}(x) \subseteq X_{\succ z}$. Let $\epsilon = \min\{\epsilon_3, \epsilon_4\}$. □

(iv) $\implies$ (i)

Proof. Consider two convergence sequences $x_n \to x$ and $y_n \to y$, where $x_n \succsim y_n$ for all n. Suppose $\neg(x \succsim y)$, which implies $y \succ x$. Then there exists an $\epsilon > 0$ such that for all $x' \in B_\epsilon(x)$ and $y' \in B_\epsilon(y)$, $y' \succ x'$. Since $x_n \to x$ and $y_n \to y$, there exists a $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in B_\epsilon(x)$ and $y_n \in B_\epsilon(y)$ such that $y_n \succ x_n$, which is a contradiction. □