ECON0106 Tutorial 2 **2026-2027**

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Overview of Section 2

We defined what it means for a preference relation to be continuous. By Debreu's Theorem, if the choice set admits a countable order-dense subset, then a continuous preference relation admits a continuous utility representation.

We can represent choices that are $\text{argmax}_{\succsim} A$ as $\text{argmax}_{x \in A} u(x)$. If the underlying preference relation is continuous, by the Weierstrass Theorem, we can guarantee that the maximum and the minimum exist.

We proved a baby version of Debreu's Theorem by considering a convex metric space on (X, d). We took 3 steps:

1. Show for all $x \succ y$, there exists $z \in Z$ such that $x \succ z \succ y$.
   (a) Convexity and openness of $\succ$ implies that there is some convex mixture x_α that has $x \succ x_\alpha \succ y$.
   (b) By subset-denseness, we can find a point $z \in Z$, such that $x \succ x_\alpha \succsim z \succ y$.
2. Define a utility representation $u : X \to \mathbb{R}$ such that $u(Z)$ is dense in $[0, 1]$.
   (a) The iterative procedure that uses the midpoints of the defined numbers.
   (b) The defined set of values is called dyadic numbers, which are dense in $[0, 1]$.
   (c) Extend u to X by taking $\sup\{u(z) | z \in Z \text{ and } x \succ z\}$.
3. Continuity of u follows from denseness and step 1.

Some more properties:

- Convexity - Deals with uniqueness of maximiser.
- Monotonicity and Insatiability - More is better.
- Homotheticity - Gives rise to representative households.
- Separability - Allows us to think about dimensions of choices that are context independent.
- Quasilinearity - Characterises "money is good", "no wealth effects", "money can compensate".

Overview of Section 3

Under some consistency conditions on choices, we can represent preferences with a utility function, and characterise preference-maximising choices as utility-maximising ones. This gives rise to the following objects:

- $x(p, w) = \text{argmax}_{x \in B(p,w)}\, u(x)$
- $v(p, w) = \sup_{x \in B(p,w)}\, u(x)$

Proposition 2 tells us v is quasiconvex in (p, w), weakly decreasing in p, and weakly increasing in w. Proposition 3 tells us that v and x are homogeneous of degree 0.

Continuity

Continuity of preferences gives us that the set of maximisers for a given wage and price system are non-empty (proposition 4), uhc and compact valued (proposition 6). Furthermore, the continuous utility representation has the property that its maximum is continuous in the wage and price system (proposition 6).

Convexity

Proposition 7 tells us that convexity of preferences allows the set of maximisers of to be convex as well.

Homotheticity

Homotheticity allows us to aggregate individual choices with a representative household (proposition 10).

Duality

The dual problem of the utility maxmisation problem is the expenditure minimisation problem:

- $h(p, u) :=_{x \in X | u(x) \geq u}\, px$ (Hicksian Demand)
- $e(p, u) := \inf_{x \in X | u(x) \geq u}\, px$

Notice that the Hicksian demand is homogeneous of degree 0 in p and the minimised expenditure is homogeneous of degree 1 in p (proposition 11). Linearity of expenditure function implies that the minimised expenditure function is concave in p (proposition 12). Theorem 3 gives the compensated law of demand. Some more implications of continuity and convexity can be found in the lecture notes.

Exercise 3 (Section 2)

Let $\succ$ be a convex preference relation on a convex set X. Then for any convex $A \in 2^X$, $\operatorname{argmax}_{\succsim} A$ is convex.

Proof. Suppose not. If $\operatorname{argmax}_{\succsim} A$ is a singleton, then it is convex, a contradiction. If $\operatorname{argmax}_{\succsim} A$ is not a singleton, then there exist some $\alpha \in (0,1)$, such that $y \succ x_\alpha = \alpha x + (1-\alpha)x'$, where $x, x' \in \operatorname{argmax}_{\succsim} A$ and $y \in A$. By proposition 1, $X_{\succsim y}$ is convex, and $x, x' \in X_{\succsim y}$, meaning that $x_\alpha \in X_{\succsim y}$. This is a contradiction. □

Let $\succ$ be a strictly convex preference relation on a convex set X. Then for any convex $A \in 2^X$, $\operatorname{argmax}_{\succsim} A$ contains at most one element.

Proof. Suppose not. Take $x, x' \in \operatorname{argmax}_{\succsim} A$. Strict concavity implies that there is some $\alpha \in [0,1]$ such that $\alpha x + (1-\alpha)x =: x_\alpha \succ x$. Since $x \succsim y$ for all $y \in A$, then $x_\alpha \succ x \succsim y$, which means that x is not an argmax. □

Exercise 2 (Section 3)

Proof. (i) This is true as $0 \in B$.

(ii) Take any $x^n \to x$ such that $x_n \in B(p,w)$, there is $\sum_i p_i x_i^n \leq w$. Weak inequality is preserved in the limit, and px is continuous in x. Hence $x \in B(p,w)$.

By definition $x \geq 0$ and $x_i \leq \max_i\{w/p_i\}$ where $p \in \mathbb{R}^k_{++}$.

Since $B \subseteq \mathbb{R}^n$ is real-valued, by Heine-Borel, $B(w,p)$ is compact.

(iii) WTS for any sequence $(p^n, w^n) \to (p_0, w_0)$, and sequence x^n such that $x^n \in B(p^n, w^n)$, there exists a subsequence $\{x^m\} \subset \{x^n\}$ such that $x^m \to x$ for $x \in B(p_0, w_0)$.

Recall that a sequence in (R^k, d_2) converges iff it converges in each dimension. Since w^n, p^n are converging sequences, they have a bounded range, i.e., $\max\{w^1, w^2, ...\} \leq W$ and $\inf\{p_i^1, p_i^2, ...\} \geq P_i > 0$. For each n, there is $x_i^n \leq \max_i \frac{w^n}{p_i^n} \leq \max_i \frac{W}{P_i}$. Hence, x^n is bounded. By the Bolzano-Weierstrass Theorem, x_n has a convergent subsequence with limit x.

Now we show that $x \in B(p_0, w_0)$. Again as weak inequalities in $\mathbb{R}^k$ are preserved in the limit and px is a continuous function of x, therefore, $p_0 x \leq w_0$.

(iv) Take any U such that $B(p_0, w_0) \cap U \neq \emptyset$. This means that there exists $\tilde{x} \in B(p_0, w_0) \cap U \neq \emptyset$. Suppose $\tilde{x} \neq 0$. As U is open, there exists some $\delta > 0$ such that $B_\epsilon(\tilde{x}) \subseteq U$. Take an element y from this open ball that satisfy the following: $y = x - \delta$. Notice that $p_0 y = p_0(\tilde{x} - \delta) < p_0 \tilde{x} \leq w_0$. Hence $y \in B(p_0, w_0) \cap U$. Define $\tilde{\delta} := w_0 - p_0 y$ and choose any ϵ such that $\epsilon y + \epsilon < \tilde{\delta}$. We can now verify that for all $(p,w) \in B_\epsilon(p_0, w_0)$ that $py \leq w$. Indeed, as $(p_0+\epsilon)y = p_0 y + \epsilon y \leq w_0 - \tilde{\delta} + \tilde{\delta} - \epsilon = w_0 - \epsilon$. If $\tilde{x} = 0$, then $x \in B(p,w)$ for all (p,w) and any $0 \in U$.

(v) Since $\mathbb{R}^k$ has a countable order-dense subset $\mathbb{Q}^k$. By Debreu's Theorem $\succsim$ admits a continuous utility representation.

(vi) Since the objective function $u(x,(w,p))$ is constant in (p,w) and continuous in x, u is continuous in $(x,(p,w))$. $B(p,w)$ is non-empty and compact valued, as well as both uhc and lhc (continuous). Hence, by Berge's Maximum Theorem, $v(p,w)$ is continuous and $x(p,w)$ is non-empty, uhc and compact-valued at all (p,w). □