**ECON0106 Tutorial 4** **2026-2027**

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## Brief Feedback for Test 1

Question 2 can be found in Manzini and Mariotti (2007).

## Overview of Section 4

In the previous lectures, we have characterised some useful properties of the solution to an optimisation problem, which we use to conceptualise individual behaviours:

$$X(S; f) := \underset{x \in S}{\operatorname{argmax}} f(x)$$

Section 4 provides methods for analysing how the behaviours change when some exogenous parameter(environment) changes, without using calculus-based approaches.

We are interested in how the solution to our optimisation problem changes, when the feasible set $S$ and the objective function $f$ change in a meaningful manner.

For ordering sets, we define the notion of strong set order. It loosely means that if a set $S$ strong set dominates $S'$, then when you take an element from each $S$ and $S'$, the *smallest* element that is simultaneously "larger" than both elements lives in $S$, and the *largest* element that is simultaneously "smaller" than both lives in $S'$.

For ordering functions, we define two ordinal properties, i.e., the single-crossing property and quasisupermodularity. We also define two cardinal properties, i.e., the increasing difference property and supermodularity. The latter is a strengthening of the former.

These tools allow us to interpret Milgrom and Shannon's monotonicity theorem.

## Exercise 1 (Section 4)

### (i)

Prove that $\neg(B(p,w) \geq_{ss} B(p,w)), \forall w > 0$.

*Proof.* To show that $B(p,w) \geq_{ss} B(p,w) \forall w > 0$ is not true, we need to find a counterexample.

Since $w > 0$ and $k \geq 2$, $\{x \in \mathbb{R}^k_+ \mid px = w\}$ is non-empty and non-singleton. Take any distinct $x, x'$ such that $px = w$ and $px' = w$. This implies that there exists some $i, j \leq k$ such that $x_i < x'_i$ and $x_j > x'_j$. Notice that $x \wedge x' \geq x$. As $px = w$, $x \wedge x' \notin B(p,w)$. □

An immediate implication is that $B(p, w)$ is not a lattice.

**(ii) Fix $p$ and provide necessary and sufficient conditions on $w$ and $w'$ so that $B(p, w') \geq_{ss} B(p, w)$.**

Fix $p$, we show that $w = 0$, $w' \geq 0$ is a necessary and sufficient condition for $B(p, w') \geq_{ss} B(p, w)$.

*Proof.* $\implies$

Since $w = 0$, then $B(p, w) = 0$. Take any $x \in B(p, w')$. It is immediate that $x \wedge 0 = x \in B(p, w')$ and $x \vee 0 = 0 \in B(p, w)$.

$\impliedby$

We prove the contrapositive: $\neg(w = 0, w' \geq 0) \implies \neg(B(p, w') \geq_{ss} B(p, w))$. Suppose otherwise. In other words, $w > 0$ and $w' \geq 0$ implies $B(p, w') \geq_{ss} B(p, w)$. Take $x$ such that $x_i = \frac{w}{p_i}$ and $x_j = 0$ for all $j \neq i$. Also take $x'$ such that $x'_m = \frac{w'}{p_m}$ for $m \neq i$ and $x'_n = 0$ for all $n \neq m$. It is immediate that $x \wedge x' \geq x'$, which implies $x \wedge x' \notin B(p, w')$. This generates a contradiction. □

The implication is that the strong set order can be too demanding sometimes when the feasible set is multidimensional. In the Robinson Crusoe example in Exercise 4, we can apply Milgrom and Shannon partly because the feasible set is one-dimensional.

## Exercise 1 (i) (Section 5)

We show the following statements:

*If $\succsim$ satisfies independence and the Archimedean property, then it satisfies vNM continuity.*

*Proof.* First notice that independence directly implies

$$\alpha p + (1 - \alpha)p' \succ \alpha p' + (1 - \alpha)p' = p', \forall \alpha \in (0, 1] \tag{*}$$

We then show that independence implies that for any $p \succ p'$ and $1 \geq \alpha > \beta \geq 0$, there is $\alpha p + (1 - \alpha)p' \succ \beta p + (1 - \beta)p'$.

$$\begin{aligned}
\alpha p + (1 - \alpha)p' &= \beta p + (1 - \beta)\left[\left(\frac{\alpha - \beta}{1 - \beta}\right)p + \left(\frac{1 - \alpha}{1 - \beta}\right)p'\right] \\
&= \beta p + (1 - \beta)\left[\left(\frac{\alpha - \beta}{1 - \beta}\right)p + \left(1 - \frac{\alpha - \beta}{1 - \beta}\right)p'\right] \\
&\succ \beta p + (1 - \beta)\left[\left(\frac{\alpha - \beta}{1 - \beta}\right)p' + \left(1 - \frac{\alpha - \beta}{1 - \beta}\right)p'\right] \\
&= \beta p + (1 - \beta)p'
\end{aligned}$$

We call this the monotonicity lemma. Now we prove the statement. Take any $p, p', p'' \in \Delta(X)$ such that $p \succsim p' \succsim p''$. If $p \sim p'$ or $p' \sim p''$, then $\gamma = 1$ or 0. Then consider the case when $p \succ p' \succ p''$. Define $E := \{\gamma \in [0,1] \mid \gamma p + (1-\gamma)p'' \succsim p'\}$. This set is non-empty set as $1 \in E$. Define $\gamma^* := \inf E$. Notice that $\gamma^* > 0$, as there exists some $\beta \in (0,1)$ where $p' \succ \beta p + (1-\beta)p''$ by Archimedean property and monotonicity lemma. We show that $\gamma^* p + (1-\gamma^*)p'' \sim p'$. Suppose not, then $\gamma^* p + (1-\gamma^*)p'' \succ p' \succ p''$. Then by Archimedean property, there exists some $\gamma' \in (0,1)$ such that $\gamma'(\gamma^* p + (1-\gamma^*)p'') + (1-\gamma')p'' \succ p'$. Notice that $\gamma'\gamma^* < \gamma^*$, where we have found another element in $E$ that is strictly smaller than $\gamma^*$. This is a contradiction. □

*If $\succsim$ satisfies independence and vNM continuity, then it satisfies Archimedean property.*

*Proof.* Take any $p, p', p'' \in \Delta(X)$ such that $p \succ p' \succ p''$. By vNM continuity, there exists some $\gamma \in (0,1)$ such that $\gamma p + (1-\gamma)p'' \sim p'$. Then, by the monotonicity lemma, we can take $\alpha \in (\gamma, 1)$ and $\beta \in (0, \gamma)$. □

## References

Manzini, Paola and Marco Mariotti (2007). "Sequentially rationalizable choice". In: *American Economic Review* 97.5, pp. 1824–1839.