ECON0106 Tutorial 7 **2026-2027**

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Subjective Expected Utility (Anscombe-Aumann's Framework)

Primitives:

- Ω states of the world (generates uncertainty)
- X set of consequences/outcomes
- $f : \Omega \to \Delta(X)$ act
- $\mathcal{F} := \Delta(X)^{\Omega}$ set of acts
- $\succsim \subseteq \mathcal{F} \times \mathcal{F}$ preference relation on acts

Here is a colloquial example to aid interpretation: Deciding whether to take an umbrella

- $\Omega = \{(\text{R})\text{ainy}, (\text{S})\text{unny}\}$
- $X = \{(\text{W})\text{et}, (\text{D})\text{ry}\}$
- $\mathcal{F} = \{(\text{T})\text{ake umbrella}, (\text{N})\text{ot take umbrella}\}$

If the state is sunny, taking an umbrella or not would always be dry, i.e., $T(S) = N(S) = \delta_D$. When the state is rainy, taking an umbrella has a 50% probability of remaining dry, with complementary probability of getting wet. Not taking an umbrella would certainly get wet, i.e.,

$$T(R) = \begin{cases} \text{W} & \text{w.p. } 50\% \\ \text{D} & \text{w.p. } 50\% \end{cases}$$
$$N(R) = \text{W for sure}$$

Depending on what the decision maker thinks about the probability of it raining, she will make different decisions based on her belief $\mu \in \Delta(\Omega)$. We want a utility function to assign values only to the outcomes $u(x)$.

Theorem 1 does not get us there initially. It states that a preference relation $\succsim$ satisfies continuity and independence $\iff$ there exists a utility function $u : X \times \Omega \to \mathbb{R}$ and a uniform probability measure $\mu \in \Delta(\Omega)$ s.t. for any pair of acts f, g,

$$f \succsim g \iff \sum_{\omega} \frac{1}{|\Omega|} \sum_{x} f(\omega)(x)u(\omega, x) \geq \sum_{\omega} \frac{1}{|\Omega|} \sum_{x} g(\omega)(x)u(\omega, x)$$

To derive a state-independent utility representation, we need the preference relation to also satisfy separability:

A preference relation $\succsim$ satisfies continuity, independence, and separability $\iff$ there exists $\mu \in \Delta(\Omega)$ and $u : X \to \mathbb{R}$ s.t.

$$f \succsim g \iff \sum_{\omega} \mu(\omega) \sum_{x} f(\omega)(x)u(x) \geq \sum_{\omega} \mu(\omega) \sum_{x} g(\omega)(x)u(x)$$

Moreover, u is unique up to positive affine transformation, and if $\exists\ f \succ g$, then μ is unique.

In Exercise 1 below, we show that when the preference relation satisfies continuity and independence, then separability and monotonicity are equivalent.

Stochastic Choice

Observables:

- X set of alternatives
- $\mathcal{A} := \{A \subseteq X | A \neq \emptyset\}$
- $\rho : \mathcal{A} \to \Delta(X)$ is a stochastic choice function (SCF) if $\text{supp}(\rho(A)) \subseteq A$; Denote $\rho(x, A)$ as the probability of choosing x from menu A

RUM:

- $(\mathcal{U}, \pi)$ s.t. $\mathcal{U} := \{u\} \subseteq \mathbb{R}^X$, and $\pi \in \Delta(\mathcal{U})$

ρ admits a RUM representation $\iff$ there is a RUM $(\mathcal{U}, \pi)$ s.t. $\rho(x, A) = \pi(\{u \in \mathcal{U} | x \in \text{argmax}_{y \in A}\, u(y)\})$. Note that, instead of saying an item maximises a utility index, we can similarly say an item is the preferred item in terms of preference relations on X. Hence,

ρ admits a RUM representation $\iff$ there is $\pi \in \Delta(R) : \rho(x, A) = \pi(\{\succsim \in R | x = \text{argmax}_{\succsim} A\})$.

When can a ρ have a RUM representation?

$$\begin{aligned} &\text{SCF } \rho \text{ admits a RUM representation} \\ \iff &\text{SCF } \rho \text{ satisfies Block-Marschak inequalities} \\ \iff &\text{SCF } \rho \text{ satisfies ARSP} \end{aligned}$$

Note that in the lecture notes, ARSP has an interesting connection to SARP.

Exercise 1 (Section 8a)

Show that if $\succsim$ is a preference relation that satisfies independence and continuity, then separability $\iff$ monotonicity.

Proof. $\implies$

Suppose $\widetilde{f(\omega)} \succsim \widetilde{g(\omega)}$ for all non-null $\omega \in \Omega$. Since $\succsim$ satisfies independence, continuity and separability, then there exists a $\mu \in \Delta(\Omega)$ and $u : X \to \mathbb{R}$, such that

$$\sum_{\omega'} \mu(\omega') \sum_{x} f(\omega)(x)u(x) \geq \sum_{\omega'} \mu(\omega') \sum_{x} g(\omega)(x)u(x)$$
$$\sum_{x} f(\omega)(x)u(x) \geq \sum_{x} g(\omega)(x)u(x)$$

Hence,

$$\sum_{\omega} \mu(\omega) \sum_{x} f(\omega)(x)u(x) \geq \sum_{\omega} \mu(\omega) \sum_{x} g(\omega)(x)u(x)$$

which implies $f \succsim g$.

$\impliedby$

Take any $p, q \in \Delta(X)$, any $h \in \mathcal{F}$, and any non-null $\omega, \omega' \in \Omega$, such that $\tilde{p}\{\omega\}h \succsim \tilde{q}\{\omega\}h$. By monotonicity,

$$\tilde{p}\{\omega\}h \succsim \tilde{q}\{\omega\}h \iff \tilde{p} \succsim \tilde{q}$$

In particular, $\tilde{q} \succ \tilde{p} \implies \tilde{q}\{\omega\}h \succ \tilde{p}\{\omega\}h$ by monotonicity. Define $f := \tilde{p}\{\omega'\}h$, $g := \tilde{q}\{\omega'\}h$. We need to check if the condition for monotonicity is satisfied. Indeed,

$$\text{At } \omega' : \widetilde{f(\omega')} = \tilde{p} \succsim \tilde{q} = \widetilde{g(\omega')}$$
$$\text{At any } \omega'' \neq \omega' : f(\omega'') = g(\omega'') = h(\omega''), \text{so } \widetilde{f(\omega'')} \sim \widetilde{g(\omega'')}$$

By monotonicity, $\tilde{p}\{\omega'\}h \succsim \tilde{q}\{\omega'\}h$, for any ω'. □

Exercise 6 (Section 8a)

The ex-post regret is $r(\omega, f, F) = \max_{g \in F} \phi(g(\omega) - f(\omega))$. This can be interpreted as "given that ω has happened, I should have chosen g". Minmax regret associated with choosing $f \in F$ is $\rho(f, F) = \max_{\omega \in \Omega} r(r, F, \omega)$. This can be interpreted as the worst possible regret the DM could get from taking the act f. There is a choice function $C_\rho(F) := \text{argmin}_{f \in F} \rho(f, F)$, that is, the DM chooses an act that minimises the worst-case regret. This setup could also be interpreted as the DM and nature playing a sequential move game.

1.

$\succsim_\rho$ is complete, as Ω is finite, so ρ are well-defined real numbers and one of them is always $\geq$ than another.

$\succsim_\rho$ also satisfies independence.

Proof. Take $f, g, h \in F$ and any $\alpha \in (0, 1]$ s.t. $f \succsim_\rho g$. Denote $f_\alpha := \alpha f + (1-\alpha)h$ and $g_\alpha := \alpha g + (1-\alpha)h$ and suppose they are in F. Notice that

$$\begin{aligned}
\rho(f, \{f, g\}) &= \max\{\phi(0), \max_\omega \phi(g(\omega) - f(\omega))\} \\
\rho(g, \{f, g\}) &= \max\{\phi(0), \max_\omega \phi(f(\omega) - g(\omega))\} \\
\rho(f_\alpha, \{f_\alpha, g_\alpha\}) &= \max\{\phi(0), \max_\omega \phi(\alpha(g(\omega) - f(\omega)))\} \\
\rho(g_\alpha, \{f_\alpha, g_\alpha\}) &= \max\{\phi(0), \max_\omega \phi(\alpha(f(\omega) - g(\omega)))\}
\end{aligned}$$

We claim that if $\alpha > 0$, then

$$\begin{aligned}
&\max\{\phi(0), \max_\omega \phi(g(\omega) - f(\omega))\} \leq \max\{\phi(0), \max_\omega \phi(f(\omega) - g(\omega))\} \\
\implies &\max\{\phi(0), \max_\omega \phi(\alpha(g(\omega) - f(\omega)))\} \leq \max\{\phi(0), \max_\omega \phi(\alpha(f(\omega) - g(\omega)))\}
\end{aligned}$$

Suppose not, then $\max_\omega \phi(\alpha(g(\omega) - f(\omega))) > \max_\omega \phi(\alpha(f(\omega) - g(\omega)))$. This means that there is some ω such that $g(\omega) - f(\omega) > f(\omega') - g(\omega')$ for all ω', which is a contradiction. We have

$$\begin{aligned}
f \succsim_\rho g &\iff \max\{\phi(0), \max_\omega \phi(g(\omega) - f(\omega))\} \leq \max\{\phi(0), \max_\omega \phi(f(\omega) - g(\omega))\} \\
(\phi \text{ increasing and } \alpha > 0) &\iff \max\{\phi(0), \max_\omega \phi(\alpha(g(\omega) - f(\omega)))\} \leq \max\{\phi(0), \max_\omega \phi(\alpha(f(\omega) - g(\omega)))\} \\
&\iff f_\alpha \succsim_\rho g_\alpha
\end{aligned}$$

□

2.

We show that $\succsim_\rho$ does not satisfy transitivity. Then, it is not a preference relation, which means that there is no choice function that satisfies Sen's α or β. Consider the following counterexample:

$$X = \{0, 2, 4\}$$

	ω_1	ω_2	ω_3
f	0	2	4
g	2	4	0
h	4	0	2

Notice that $\rho(f,\{f,g\}) = \phi(2)$ and $\rho(g,\{f,g\}) = \phi(4)$. Hence $f \succ_\rho g$. Similarly, we have $g \succ_\rho h$ and $h \succ_\rho f$. Hence, transitivity fails.

3.

$\succsim_\rho$ satisfies uncertainty aversion.

Proof. $f \sim_\rho g \iff \rho(f,\{f,g\}) = \max\{\phi(0), \max_\omega \phi(f(\omega)-g(\omega))\} = \rho(g,\{f,g\}) = \max\{\phi(0), \max_\omega \phi(g(\omega)-f(\omega))\}$. This implies that $\max\{\phi(0), \max_\omega \phi(\frac{1}{2}(f(\omega) - g(\omega)))\} = \max\{\phi(0), \max_\omega \phi(\frac{1}{2}(g(\omega) - f(\omega)))\}$, by the same argument as in question 1. Then,

$$\begin{aligned}
\rho\left(\frac{1}{2}f + \frac{1}{2}g, \left\{\frac{1}{2}f + \frac{1}{2}g, f\right\}\right) &= \max\left\{\phi(0), \max_\omega \phi\left(\frac{1}{2}f(\omega) - \frac{1}{2}g(\omega)\right)\right\} \\
&= \max\left\{\phi(0), \max_\omega \phi\left(\frac{1}{2}(f(\omega) - g(\omega))\right)\right\} \\
&= \max\left\{\phi(0), \max_\omega \phi\left(\frac{1}{2}(g(\omega) - f(\omega))\right)\right\} \\
&= \rho\left(f, \left\{\frac{1}{2}f + \frac{1}{2}g, f\right\}\right)
\end{aligned}$$

Hence $\frac{1}{2}f + \frac{1}{2}g \succsim_\rho f$. □

4.

This is Savage 1951's minimax-regret rule. Notice that it needs no belief.

Stochastic Choice Question

Let X be a finite set of alternatives. Recall two properties of a Stochastic Choice Function (SCF) ρ:

1. (Stochastic) Monotonicity: For any $x \in B \subseteq A \subseteq X$, $\rho(x,B) \geq \rho(x,A)$.
2. Weak Stochastic Transitivity (WST): For any $x,y,z \in X$, if $\rho(x,\{x,y\}) \geq \frac{1}{2}$ and $\rho(y,\{y,z\}) \geq \frac{1}{2}$, then $\rho(x,\{x,z\}) \geq \frac{1}{2}$. Fix $X = \{x,y,z\}$. Prove or find a counterexample:

- Monotonicity implies WST.
- WST implies Monotonicity.
- Comment on what your findings imply for stochastic choice representations.

References

Savage, Leonard J (1951). "The theory of statistical decision". In: *Journal of the American Statistical association* 46.253, pp. 55–67.